Sensitivity of Complex, Internally Coupled Systems

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A method is presented for computing sensitivity derivatives with respect to independent (input) variables for complex, internally coupled systems, while avoiding the cost and inaccuracy of finite differencing performed on the entire system analysis. The method entails two alternative algorithms: the first is based on the classical implicit function theorem formulated on residuals of governing equations, and the second develops the system sensitivity equations in a new form using the partial (local) sensitivity derivatives of the output with respect to the input for each part of the system. A few application examples are presented to illustrate the discussion. The method has potential to answer the "what if" questions by presenting engineers with sensitivity information on design tradeoffs to guide human judgment and formal optimization. In addition, the method is compatible with the modern technology of distributed computing as well as traditional division of design tasks among groups of specialists in the design process. The capability to quantify the effects of proposed design changes may provide the basis for a mathematical model of design.

Nomenclature

A_i = the ith CA
CA = contributing analysis, a "black box" transforming input into output data used in analysis of a system; usually associated with an engineering discipline, or a physical part of the system
F = vector of functions forming the equations governing a physical phenomenon
f = functional relationship
GSE1 = global sensitivity equation based on the partial derivatives of the governing equation residuals
GSE2 = global sensitivity equation based on the partial derivatives of output with respect to input of each CA
H = number of input items received by a CA from other CA's (bandwidth)
I = identity matrix
M = number of independent variables in a CA
m = number of unknown variables in a CA
N = number of CA's in a system analysis. System Analysis: one solution of Eqs. (1) for all of the output unknowns Y
X = vector of independent variables
Y = vector of dependent variables
Z = number of unknown variables in a CA
α,β,γ = identifiers for CA's in a small system of three CA's, equivalent of A_1, A_2, A_3
$\bar{\ }$ = linearized function
$\tilde{\ }$ = normalized, nondimensional quantity

Subscripts

i,j,k = subscripts identifying CA's, elements of vectors, and elements of matrices
o = superscript or subscript identifying an initial value, or a normalization denominator

Received Feb. 9, 1988; presented as Paper 88-2378 at the AIAA/ASME/ASCE/AHS 29th Structures, Structural Dynamics, and Materials Conference, Williamsburg, VA, April 18–20, 1988; revision received Oct. 1, 1988. This paper is declared a work of the U.S. Government and is not subject to copyright protection in the United States.

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Introduction

"WHAT if" is the all-important question that arises again and again in design. Indeed, it may be argued that the design process is not complete until all such pertinent questions have been asked, satisfactorily answered, and the answers translated into design changes toward a product as good as it can be made under a set of given restrictions. If the object being designed is a complex, coupled system, the "what if" questions are difficult to answer because, to borrow a phrase from Ref. 2, "if you make any change to it there are likely to be many subtle consequences." A recent example from aerospace is the forebody shape in hypersonic aircraft whose change influences structures, aerodynamics, propulsion, and, ultimately, the performance.

Many "what if" questions cannot be quantified, and engineering judgment is indispensable to answer them. However, in aerospace vehicle design a great deal of "what if" questions can be quantified either by assessing the effects of relatively large variations of the variables involved (a parametric study) or by considering very small, theoretically infinitesimal variations to calculate sensitivity derivatives.

The focus of this paper is on sensitivity analysis. Although recent developments in numerical methods provided engineers with many useful techniques for disciplinary, or subsystem, sensitivity analysis, e.g., Refs. 3–5, examination of literature, e.g., Refs. 2, 6, and 7, shows a void as far as the comparable methods applicable to entire systems are concerned. The purpose of this paper is to address that void and to offer a system sensitivity analysis capable of answering the quantitative "what if" design questions. To that end, the paper presents a method for computing sensitivity derivatives with respect to independent (input) variables for complex, internally coupled systems, while avoiding the cost and inaccuracy of finite differencing performed on the entire system analysis. The method entails two alternative algorithms: the first is based on the classical implicit function theorem formulated on residuals of governing equations, and the second develops the system sensitivity equations in a new form using the partial (local) sensitivity derivatives of the output with respect to the input for each part of the system. A few application examples are presented to illustrate the discussion. To conform to the space limitations, the paper was abridged in some details that can be found in its full-length version in Ref. 1.

Statement of the Problem

In this paper, a complex, internally coupled system is defined as physical object whose behavior is described by a vector Y obtainable as a solution of a set of simultaneous (coupled) equations which can be partitioned into subsets such as

$$\alpha\ [(X, Y_\beta, Y_\gamma),\ Y_\alpha] = 0 \tag{1a}$$

$$\beta\ [(X, Y_\alpha, Y_\gamma),\ Y_\beta] = 0 \tag{1b}$$

$$\gamma\ [(X, Y_\beta, Y_\alpha),\ Y_\gamma] = 0 \tag{1c}$$

Each of the system subsets represents a distinct, separate analysis that will be referred to as contributing analysis (CA), usually associated with a particular engineering discipline, or a distinct physical part (a subsystem) of the system, or both. Partitioning of the system analysis into separate but coupled CA's amounts to a system decomposition. The operations research literature calls such partitioning an aspect decomposition if they correspond to physical subsystems.[8] In most engineering problems, both types of decomposition are used simultaneously to break the large task into smaller ones. Mathematics developed in this paper applies equally to both types. All of the mathematical discussion herein is based on three partitions because that is a number which is conveniently small and yet sufficient to establish patterns that can easily be generalized to arbitrarily large number of partitions. Solving the entire set of equations, which can be written as $F(Y,X) = 0$, will be referred to as the system analysis.

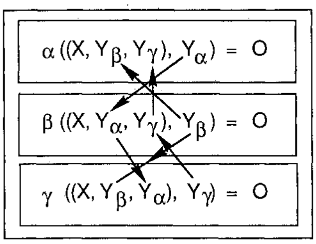

Fig. 1 Functions vectors forming a set of coupled equations.

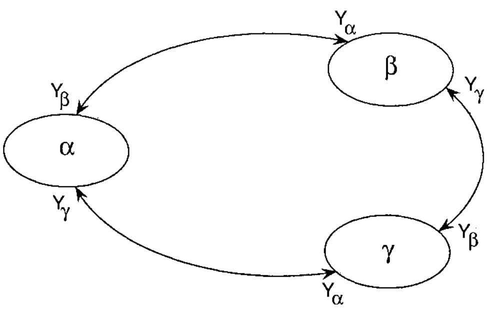

Fig. 2 Directed graph representation of the system shown in Fig. 1.

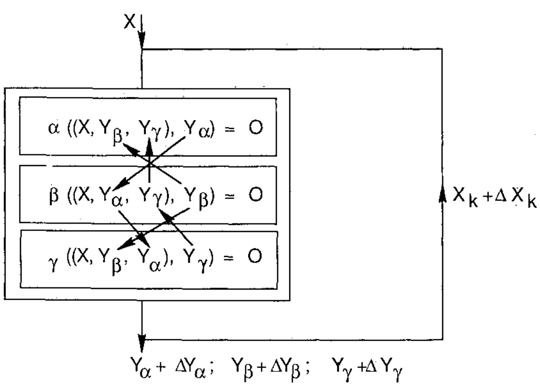

Fig. 3 Finite-difference procedure involving the system analysis.

Each CA yields a solution in form of a vector $Y_\cdot$ (where subscript $\cdot$ stands for α, β, or γ, and identifies a subset of Y) listed last in the brackets, given the input listed in the inner parentheses. The system is internally coupled because the input to one CA includes outputs from the other CA's—as shown by the arrows in Fig. 1. The coupled system is depicted in Fig. 2 by a directed graph representation (e.g., Ref. 2).

The focus of this paper is on large-scale applications in which at least some CA's are nonlinear and complex, so that the system analysis can only be done iteratively. Typically, a CA is carried out by a group of specialists, maybe at a separate subcontractor organization. This may be illustrated by an example of aircraft wing design incorporating nonlinear aerodynamics, structures, and active control (aspect decomposition), or substructuring (object decomposition).

The system solution Y is sensitive to the independent variables X present in the CA inputs. It is important to emphasize that the independent variables X may include not only the designer-decided inputs (design variables) but also other imputs external to the system, for example, loads, heat flux, etc. In the most general case, all variables X may occur in the input to each CA, but in most practical applications only a subset of the vector X will enter the input of a particular CA.

One way to compute sensitivity derivatives of the solution Y with respect to the independent variables X is a finite-difference technique depicted by a flowchart in Fig. 3 in its simplest, one-step-forward version. It requires repetition of the system analysis for every perturbed X. This may be prohibitively costly, particularly if the system analysis is nonlinear and/or iterative. Even more importantly, it may be inaccurate to the point of producing meaningless results as the effect of small perturbations in X may drown in the noise of the iterative solution of the system (e.g., Ref. 9). Attempting to remedy this effect by increasing the perturbation magnitude may introduce significant error due to the analysis nonlinearity. Consequently, the perturbation range in which accuracy of finite differencing is acceptable becomes problem dependent and may not even exist.

Thus, the problem is how to calculate the sensitivity derivatives of the system solution Y with respect to the independent variables X without resorting to a finite-difference operation involving the entire system analysis as in Fig. 3.

Solution

As mentioned in the Introduction, there are at least two ways of solving the system sensitivity problem. A residual-based solution will be introduced first, and an alternative using local output sensitivity will follow.

Residual-based Solution

The implicit function theorem of functional analysis, e.g., Ref. 10, states that a set of governing equations

$$F(Y,X) = 0 \tag{2a}$$

$$Y = f(X) \tag{2b}$$

has the following sensitivity equation

$$\left[\frac{\partial F}{\partial Y}\right] \left\{\frac{\partial Y}{\partial X_k}\right\} = -\left\{\frac{\partial F}{\partial X_k}\right\} \tag{3}$$

The sensitivity equations are always simultaneous, linear, and algebraic, regardless of the mathematical nature (nonlinear, transcendental, etc.) of the governing equations of the system. In Eq. (3), the matrix of coefficients, $m \times m$, is a Jacobian matrix of the partial derivatives with respect to dependent variables, and the right-hand-side vector contains the partial derivatives with respect to a particular independent variable.

These partial derivatives are evaluated using the X and Y values which satisfy Eqs. (2). In other words, a solution Y of the governing equations, Eqs. (2), is a prerequisite to forming and solving the sensitivity equation, Eq. (3).

The solution vector of Eq. (3) comprises the derivatives of the dependent variables with respect to a particular independent variable. It will be useful in further discussion to have noted at this point that Eq. (3) is based on residuals of Eqs. (2), i.e., a perturbation of one element in X alone would generate a vector of residuals of F replacing zero on the right-hand side of the equation. Similarly, a perturbation of one element in Y alone would also generate a residual vector. Consequently, to maintain the right-hand side at zero despite the perturbation of X, there must be a change in Y subordinated to the change of X to make the residual vectors due to Y and X offset each other. Equation (3) merely states that to generate compensating residuals the rates of change of the residuals with respect to the dependent and independent variables must balance each other, taking into account the implicit dependence of Y and X. In other words, the total derivative with respect to X of the residuals of Eqs. (2) must vanish.

The method for computing the terms of Eq. (3) is problem dependent. Obviously, an analytical differentiation is preferred but, if that is not possible, a finite-difference technique may be applied. Since the finite-difference technique in this application is used to calculate the partial derivatives of residuals, it requires only an evaluation of $F(Y,X)$ for perturbations of X and Y instead of a solution of $F(Y,X) = 0$ for each perturbation of X. Thus, the finite-difference operation performed on the entire system analysis as in Fig. 3 is eliminated.

When applied to the partitioned system in Eq. (1), the sensitivity equation, Eq. (3), takes on this form

$$\begin{bmatrix} \partial\alpha_i/\partial Y_{\alpha j} & \partial\alpha_i/\partial Y_{\beta j} \\ \partial\beta_i/\partial Y_{\alpha j} & \partial\beta_i/\partial Y_{\beta j} \\ \partial\gamma_i/\partial Y_{\alpha j} & \partial\gamma_i/\partial Y_{\beta j} \end{bmatrix} \begin{Bmatrix} \partial\alpha_i/\partial Y_{\gamma j} & \partial Y_{\alpha i}/\partial X_k \\ \partial\beta_i/\partial Y_{\gamma j} & \partial Y_{\beta i}/\partial X_k \\ \partial\gamma_i/\partial Y_{\gamma j} & \partial Y_{\gamma i}/\partial X_k \end{Bmatrix}$$

$$= - \begin{Bmatrix} \partial\alpha/\partial X_k \\ \partial\beta/\partial X_k \\ \partial\gamma/\partial X_k \end{Bmatrix} \tag{4}$$

and is referred to as the global sensitivity equation 1 (GSE1). These equations contain as unknowns the sensitivity derivatives of the system solution Y (partitioned) with respect to an independent variable X (one at a time). Their matrix of coefficients is populated by the partial derivatives of the residuals of each CA with respect to the input that CA receives from the other CA's, and the right-hand-side vector is formed from the partial derivatives of the CA residuals with respect to the independent variable directly affecting that CA. For a general case of N CA's, the equations acquire a format shown in the Appendix.

Despite their potential cost and accuracy advantages, the use of the sensitivity equation (4) based on residuals may not be straightforward in engineering practice because existing disciplinary codes have usually no provisions to compute the residuals, and the residuals usually have no obvious physical meaning that would allow the user to judge validity of the numbers. (An exception is structural analysis where residuals are unequilibrated loads.) These reasons motivated derivation of a new form for the system sensitivity equations not predicated on the residuals.

Formulation Based on Sensitivities of Individual CA's

Residual-independent sensitivity equations may be derived in more than one way. The derivation shown below is based on linearization of the governing equations, Eqs. (1); an alternative derivation is shown in the Appendix.

Equations (1a–1c) relate each partition of Y to X and the other partitions of Y so that from each equation

$$Y_\alpha = f_\alpha\,(X, Y_\beta, Y_\gamma) \tag{5a}$$

$$Y_\beta = f_\beta\,(X, Y_\alpha, Y_\gamma) \tag{5b}$$

$$Y_\gamma = f_\gamma\,(X, Y_\beta, Y_\alpha) \tag{5c}$$

These functions may be linearized in the neighborhood of the solution of Eqs. (1) denoted as $Y_{\alpha o}$, $Y_{\beta o}$, and $Y_{\gamma o}$ using a Taylor series abridged to its linear part. Using Y_α as an example,

$$Y_\alpha = Y_{\alpha o} + (\partial f_\alpha/\partial X)\,\Delta X + (\partial f_\alpha/\partial Y_\beta)\,\Delta Y_\beta + (\partial f_\alpha/\partial Y_\gamma)\,\Delta Y_\gamma \tag{6}$$

By moving all terms to the left-hand side, Eq. (6) is transformed into a linearized version of Eqs. (1):

$$\begin{aligned} \bar{\alpha} = Y_\alpha - Y_{\alpha o} &- (\partial f_\alpha/\partial X)(X - X_o) \\ &- (\partial f_\alpha/\partial Y_\beta)(Y_\beta - Y_{\beta o}) \\ &- (\partial f_\alpha/\partial Y_\gamma)(Y_\gamma - Y_{\gamma o}) = 0 \end{aligned} \tag{7a}$$

$$\begin{aligned} \bar{\beta} = Y_\beta - Y_{\beta o} &- (\partial f_\beta/\partial X)(X - X_o) \\ &- (\partial f_\beta/\partial Y_\alpha)(Y_\alpha - Y_{\alpha o}) \\ &- (\partial f_\beta/\partial Y_\gamma)(Y_\gamma - Y_{\gamma o}) = 0 \end{aligned} \tag{7b}$$

$$\begin{aligned} \bar{\gamma} = Y_\gamma - Y_{\gamma o} &- (\partial f_\gamma/\partial X)(X - X_o) \\ &- (\partial f_\gamma/\partial Y_\alpha)(Y_\alpha - Y_{\alpha o}) \\ &- (\partial f_\gamma/\partial Y_\beta)(Y_\beta - Y_{\beta o}) = 0 \end{aligned} \tag{7c}$$

Under some conditions discussed in the Appendix, Eqs. (7) may have no solution because of singularity of their matrix of coefficients. Assuming that singularity conditions examined in the Appendix do not occur, it is axiomatic that Eqs. (7) and (1) have the same solution Y and that this solution has the same derivatives with respect to X; consequently, we may treat Eqs. (7) as surrogate governing equations. The implicit function theorem may now be applied to these equations just as it was applied to Eqs. (1) by performing the differentiation shown in Eq. (3). This yields sensitivity equation in the form:

$$\begin{bmatrix} I & -\partial f_\alpha/\partial Y_\beta & -\partial f_\alpha/\partial Y_\gamma \\ -\partial f_\beta/\partial Y_\alpha & I & -\partial f_\beta/\partial Y_\gamma \\ -\partial f_\gamma/\partial Y_\alpha & -\partial f_\gamma/\partial Y_\beta & I \end{bmatrix}$$

$$\begin{Bmatrix} \partial Y_\alpha/\partial X_k \\ \partial Y_\beta/\partial X_k \\ \partial Y_\gamma/\partial X_k \end{Bmatrix} = \begin{Bmatrix} \partial f_\alpha/\partial X_k \\ \partial f_\alpha/\partial X_k \\ \partial f_\alpha/\partial X_k \end{Bmatrix} \tag{8}$$

termed global sensitivity equations 2 (GSE2). For a general case of N CA's, the equations acquire a format shown in the Appendix.

Equation (8) contains no residuals of the CA's. Instead, its matrix of coefficients is populated by the senstivity derivatives of each CA output with respect to that CA's input, and its right-hand-side vector represents sensitivity of a CA's output with respect to the independent variable (one at a time) directly affecting that CA. As far as the complete system is concerned, these derivatives are partial (local) derivatives, while the solution of Eq. (8) yields the derivatives of the

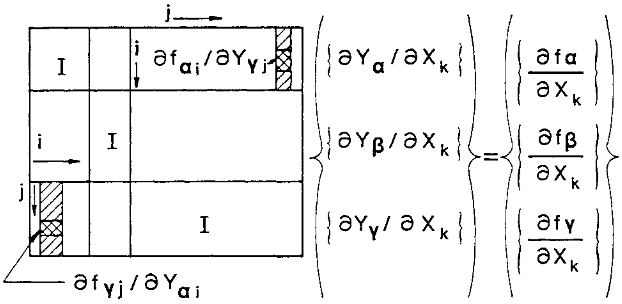

Fig. 4 Anatomy of the GSE2.

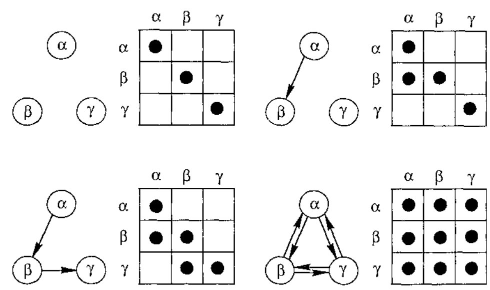

Fig. 5 System couplings reflected in the GSE2 matrix.

solution Y (partitioned) with respect to an independent variable (one at a time). By definition, the partial derivates represent senstivity of each isolated CA, and the derivatives of Y represent the system sensitivity with all of the couplings (e.g., Figs. 1 and 2) fully accounted for.

Both GSE1 and GSE2, Eqs. (4) and (8), produce the same solution vector and both are equally exact because they are derived from a mathematical theorem without any simplifying assumptions or approximations. Similarity of the two sets of equations in their mathematical characteristics and potential usage allows one to limit the ensuing discussion to GSE2, occasionally using the notation GSEx to address both GSE1 and GSE2. This emphasis on GSE2 does not imply an unqualified recommendation of GSE2 over GSE1. The choice is up to the user and it depends on the factors already stated in the foregoing as motivations for the GSE2 development, and on the considerations of cost and benefits discussed later.

Figure 4 illustrates details of the GSE2 structure for three CA's. The general pattern is easily extrapolated to larger number of CA's as shown in the Appendix. The matrix of coefficients has identity submatrices on the diagonal. Each off-diagonal submatrix is a Jacobian matrix corresponding to a CA. For example, the Jacobian at the upper-right corner in Fig. 4 contains partial sensitivity derivatives of every item of output from the α CA, a column m_α long, with respect to every one of m_γ input items the α CA receives from the γ CA, hence the m_γ columns in the Jacobian. The corresponding Jacobian at the lower-left corner comprises the partial derivatives of the output from the γ CA with respect to the input the γ CA receives from the α CA. In the general case, the two Jacobians are not symmetric. An example of the above two Jacobian matrices might be drawn from a case of an actively controlled flexible wing. Then, assuming the CA's α, β, and γ to be aerodynamics, structures, and active controls, respectively, the upper-right Jacobian would include the partial sensitivity derivatives the aerodynamic pressure at selected locations on the wing to the control surface deflections. Correspondingly, the lower-left Jacobian might comprise of the partial derivatives of the control surface deflections to the aerodynamic pressure coefficients. These derivatives derive from the control law that establishes a functional relationship between the control surface deflections and the aerodynamic pressure on the wing (sensed directly or indirectly).

On the right-hand side, there is a vector of partial sensitivity derivatives of the CA output with respect to a particular independent variable X. These partial derivatives are nonzero for those CA's that are directly influenced by that particluar independent variable. Referring again to the above example of a flexible wing, if the variable X were, say, the planform aspect ratio, the nonzero elements of the right-hand-side vector would occur at the location corresponding to the β and γ partitions, since only the aerodynamics and structures would be directly influenced.

The GSE2 matrix of coefficients depends only on the coupling among the CA's and not at all on the sensitivity to the design variables. The opposite is true for the right-hand-side vectors. Thus, the matrix can be formed and factored once, and the solutions for many design variables can be obtained by repeated back-substitutions of each right-hand-side vector. The coupling among the CA's is reflected in the topology of the GSE2 matrix as shown in a few examples in Fig. 5. If there are no couplings, the matrix is an identity matrix and the derivatives of Y are equal directly to the partial derivatives on the right-hand-side. Each coupling link generates an off-diagonal Jacobian until the matrix becomes fully populated for a fully coupled system. The coefficient matrix in the GSE2 exhibits the same pattern.

There is a coincidence of form between the matrix of coefficients in GSEx and the so-called equation precedence matrix (or N-square matrix) used in operations research literature [e.g., (Ref. 2, p. 87)] to analyze internal couplings in systems; namely, each nonzero, off-diagonal Jacobian in the GSEx matrix of coefficients corresponds to a nonzero element in the N-square matrix for the same system.

As a matter of a particular interest to a structural engineer, one may observe that if F in Eq. (2) represents structural load-deflection equations, then the Jacobians in Eq. (8) correspond to substructures or, ultimately, individual finite elements.

In some applications, it may be convenient (for instance, when the X variables are measured in different units) to have all terms in GSE2 dimensionless. A nondimensional version of the GSE2 is given in the Appendix.

Examples

Since both the GSE1 and GSE2 [Eqs. (4) and (8), respectively] are rigorously derived from a fundamental theorem, they do not need numerical verification. However, a few examples are provided for the usage of GSE2 to support the discussion of costs and benefits that will follow.

A simple example of a two-dimensional airfoil in airflow is shown in Fig. 6. The airfoil is supported by two linear springs attached to a ramp whose angle of inclination Ψ is an independent variable. The elastic degrees of freedom allowed are only pitch and plunge. The lift coefficient is assumed to be a nonlinear function of the angle of attack illustrated in Fig. 7 and defined in Table 1—Aerodynamics. The function is set up deliberately as a transcendental function to admit only an iterative system analysis. The angle of attack θ depends on the ramp angle (design variable Ψ and the airfoil elastic support pitch angle ϕ).

The airfoil on springs is an aerodynamics-structure system abstracted as a directed graph in Fig. 8. All of the equations that constitute the aerodynamics and structures CA's in the graph are listed in Table 1, which also shows the problem notation and its correspondence to the generic notation used in the paper, and the numerical data for the example. The purpose of the example is to show computation of the derivatives of the system solution output—the lift L and the elastic pitch angle ϕ—by means of the GSE2 and to compare the results with those from a finite-difference techinque.

The system solution was found iteratively and is listed in Table 2 for arbitrary Ψ value of 0.05 rad. Next, the sensitivity derivatives of L and Ψ with respect to the angle Ψ were obtained by the finite-differences procedure at the system level illustrated in Fig. 3, which required repetition of the iterative solution for the angle Ψ incremented by 0.0025 rad to 0.0525 rad. These derivatives are shown in Table 2 and provide reference for comparsion with the same derivatives computed using the GSE2. Incidentally, the derivative of L is greater than the partial derivative due to the elastic effect. The GSE2 and the numerical values of the partial derivatives that enter these equations are also given in Table 2 (these equations are also shown in a dimensionless format defined in the Appendix). The partial derivatives were obtained by the same, simple, one-step-forward, finite-difference procedure referred to above but applied separately to aerodynamics and structures CA's. Finally, Table 2 presents the GSE2 solutions that agrees with the finite-difference results obtained at the system level.

The second example shows how the GSE2 equations for a system are made up of the partial derivatives for the system CA's. The system is a flexible wing with an active control intended to reduce the root bending moment. The system directed graph and the coupling information are shown in the upper part of Fig. 9. The bottom part of the figure illustrates the make-up of the GSE2.

Dimensions of the arrays entering the GSE2 depend on the number of individual pieces of data (coupling channel bandwidth, referred to as bandwidth, for short) communicated from one CA to another. These dimensions have a strong impact on the computational cost of the method as shown in the next section; therefore, it is important to keep the bandwidths as small as possible. In this example, the structures-active control channel does not need to transmit more than a few strain-gage readings. Similarly, the aerodynamics-active control channel transmits only a few dynamic pressure sensor indications (or only the Mach number and the angle-of-attack value from which the pressure may be inferred) and one, or two, control-surface deflection angles. In contrast, the information moving along the aerodynamic-structures channel may include hundreds of the dynamic pressure values for discrete locations on the wing, if a panel-based CFD code is used, and thousands of the nodal point displacements output from a finite-element code. It is evident that this channel will require attention to reduce its bandwidth. Such reduction may be achieved by representing deformations and loads by a relatively small number of generalized coordinates and corresponding generalized forces based on modal analysis, following the practice well-established in aeroelasticity analysis.

Another example of the use of the GSE2 for a system with active control is described in (Ref. 9).

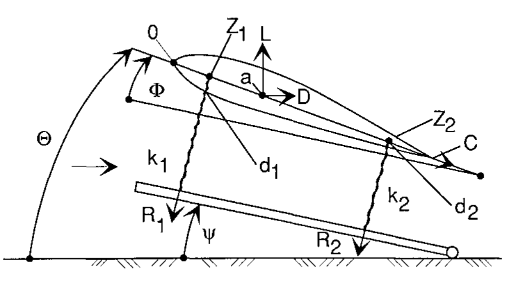

Fig. 6 Example 1: a simple aerodynamics-structures system.

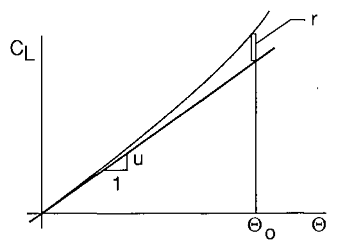

Fig. 7 Nonlinear relationship C_L vs angle of attack in example 1.

Costs and Benefits

By using GSEx [Eqs. (4) or (8)], the cost of repetitive system analysis required by the finite-difference procedure (Fig. 3) is eliminated, but the cost of generating the input into these equations and solving them is added. Using the CPU time as a simplified measure of the computational cost, Ref. 1, Appendix B, shows that, under a set of assumptions, the cost for the finite-difference procedure of Fig. 3 increases with N^2. On the other hand, the cost of generating the input into GSEx

Table 1 Definition of example 1

Notation and data

Notation: b — span
C — chord

$\bar{z}_1 \equiv z_1/C, \quad \bar{z}_2 \equiv \bar{z}_2/C, \quad \bar{a} = a/C$

$\bar{h}_1 \equiv \bar{a} - \bar{z}_1, \quad \bar{h}_2 \equiv \bar{z}_2 - \bar{a}$

$p \equiv \bar{h}_1/\bar{h}_2$

$S = B \cdot C$

$Y_\alpha = \{L, c, \bar{a}\}, \quad X_\alpha = \{S, u, r, \theta_o, c, \bar{\alpha}, \Psi)$

$L = f_\alpha(\Psi, \phi), \quad \phi = f_\beta(L)$

$Y_\beta = \{\phi\}, \quad X_\beta = \{\bar{z}_1, \bar{z}_2, k_1, k_2\}$

Data: $B = 100$ cm, $C = 10$ cm, $\bar{z}_1 = 0.2$; $\bar{z}_2 = 0.7$

$k_1 = 4000$ N/cm, $k_2 = 2000$ N/cm

$\bar{a} = 0.25, \quad q = 1$ N/cm^2, $\theta_o = 0.26$ rad

Coupling data moving down	Aerodynamic CA	Coupling data moving up
	$L = q \cdot S \cdot C_L$	
	$\theta = \phi + \psi, \quad (\#)$	
	$C_L = u\theta + r\{1 - \cos[(\pi/2)(\theta/\theta_o)]\}$	
L	Structural CA	ϕ
	$R_1 = L/(1+p)$	
	$R_2 = Lp/(1+p)$	
	$d_1 = R_1/k_1$	
	$d_2 = R_2/k_2$	
	$\phi = (d_1 - d_2)/[C \cdot (\bar{z}_2 - \bar{z}_1)]$	

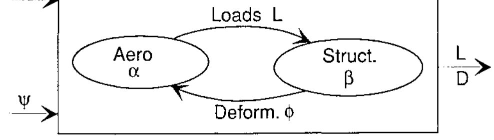

Fig. 8 The system from Fig. 6 abstracted as a "black box" with a directed graph showing internal coupling.

Table 2 Sensitivity analysis of example 1

System solution for $\Psi = 0.05$ rad

$$L = 502.3\ N;\ \phi = 0.0176 \text{ rad}$$

Derivatives with respect to Ψ by finite differences

$\Delta\Psi = 0.0025$

$$\frac{\partial L}{\partial \Psi} = 14925.16\ N/\text{rad};\ \frac{\partial \phi}{\partial \Psi} = 0.5221287 \text{ rad/rad}$$

GSE2

Symbolically:

$$\begin{bmatrix} I & -\partial f_\alpha/\partial\phi \\ -\partial f_\beta/\partial L/ & I \end{bmatrix} \begin{Bmatrix} \partial L/\partial\Psi \\ \partial\phi/\partial\Psi \end{Bmatrix} = \begin{Bmatrix} \partial f_\alpha/\partial\Psi \\ 0 \end{Bmatrix}$$

Numerically:

$$\begin{bmatrix} 1 & -9805.105 \\ -0.3510^{-4} & 1 \end{bmatrix} \begin{Bmatrix} \partial L/\partial\Psi \\ \partial\phi/\partial\Psi \end{Bmatrix} = \begin{Bmatrix} 9805.105 \\ 0 \end{Bmatrix}^{a}$$

Numerically, in a dimensionless format per Eqs. (A5) and (A6)

$$\begin{bmatrix} I & -0.343 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \partial \bar{L}/\partial\bar{\Psi} \\ \partial\bar{\phi}/\partial\bar{\Psi} \end{Bmatrix} = \begin{Bmatrix} 0.343 \\ 0 \end{Bmatrix}$$

Derivatives with respect to Ψ from GSE2

$$\frac{\partial L}{\partial \Psi} = 14928.12\ N/\text{rad};\ \frac{\partial \phi}{\partial \Psi} = 0.5224841 \text{ rad/rad}$$

[a]Numerical values of $\partial f_\alpha/\partial\phi$ and $\partial f_\alpha/\partial\Psi$ are equal because of relation marked # in Table 1.

under the same set of assumptions is proportional to the product of N and H. Proportionality to N^2 indicates that the finite-difference procedure cost may tend toward overwhelming values for large N, thus precluding the procedure applicability to large systems. Correspondingly, the proportionality to NH forecasts that the cost advantage of the GSEx-based sensitivity analysis over the finite-difference procedure will increase with the size of the system measured by N, provided that the bandwidth magnitude H is judiciously kept under control and the cost of solving the block-sparse equations GSEx is moderated by taking advantage of the progress in computing technology. Reference 1, Appendix B, shows also that the GSE1 computational cost is likely to be at least an order of magnitude smaller than that of GSE2 because evaluation of the partial derivatives in each CA is replaced by much less expensive evaluation of the residuals.

As far as the accuracy of GSEx is concerned, it depends on the conditioning of its matrix of coefficients (see Appendix) but is not affected by the system dimensionality.

The principal qualitative advantage of the sensitivity analysis based on the GSE2 lies in its providing a means to treat the system as decomposed into a set of "black boxes" coupled by well-defined sets of data. Each black box may, then, be subjected to its own sensitivity analysis performed by specialists intimately familiar with the specifics. The specialists may use any means for the partial sensitivity analysis available such as finite-difference procedures, historical statistical data, approximate methods, or even judgmental assessment. It should also be stressed that they may also draw on the disciplinary, quasianalytical sensitivity analysis algorithms that are now undergoing an intensive development.[7] They may even obtain the sensitivity data experimentally. In general, the approach

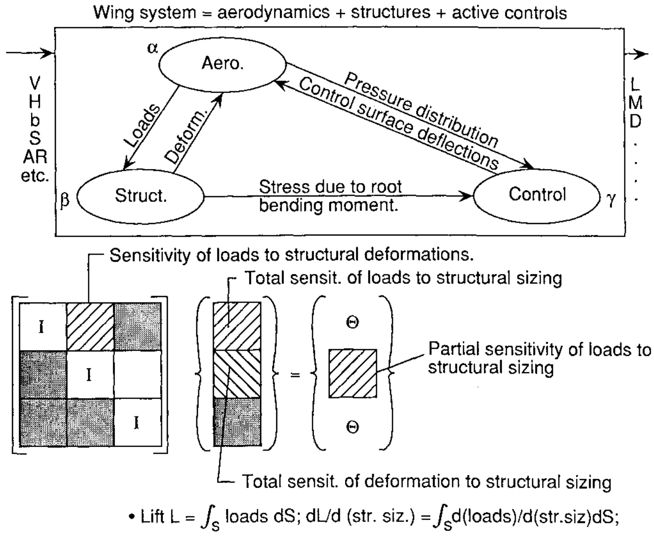

Fig. 9 Example of a flexible wing: a system comprising aerodynamics, structures, and active control.

divides the labor and thus creates opportunity for concurrent data processing in the contemporary, distributed computing environment, and supports a broad workfront in the engineering organization.

Another benefit from the "black-box" approach is that the GSEx are inherently recursive, in the sense that each of the system's "black boxes" (the CA's) may be a complex system within itself. If so, its sensitivity analysis may be carried out as described herein, to produce the sensitivity derivatives that will be treated as the sensitivity partial derivatives in the GSEx of the parent system.

Regarding the choice of GSE2 vs GSE1 if the computational cost was the only factor (Ref. 1, Appendix B), GSE1 would be recommended over GSE2. However, the nonavailability of the residuals in existing disciplinary codes, and difficulties with physical interpretation of the residuals clearly favor the GSE2 format. Furthermore, the disciplinary sensitivity analyses are formulated to yield data compatible with input to GSE2 but not GSE1. These considerations may be overridden in the future by new disciplinary code developments (availability of the residual as an option) encouraged by the strong cost advantage of the GSE1. For now, the choice is judgmental.

Use in Design

Systematic procedure for using the GSE2 in a design process may be organized in the following steps:

1) Analyze the system for a given X to obtain Y.

2) For the given X and for the Y obtained above, calculate the partial derivatives [Eq. (8), also Eq. (A4)] in each CA separately. These calculations may be performed concurrently.

3) Assemble and solve GSE2 for the system sensitivity derivatives.

4) Use the system senstivity derivatives to decide how to change X in order to improve the system behavior.

5) Repeat, from step 1 with new X, and continue until a satisfactory system behavior is attained.

The procedure for the GSE1 will differ only in the type of the partial derivatives computed in step 2. In step 4, one may use the system sensitivity derivatives information in a number of ways, e.g., to reduce the number of variables X by identifying the most influential ones ("the design drivers"), to support judgmental redesign decisions, and to use that information as input into a formal optimization algorithm. In the general case, the system solution and its sensitivity derivatives have to be updated after moving away from the previously solved design, so the procedure has to be iterated as indicated in step 5.

The GSEx may also be used to assess the coupling strength between any two CA's. This can be done by computing derivatives of the GSEx solution to the elements of the GSEx matrix in a manner shown below for the GSE2 used as an example.

The GSE2, Eqs. (8) may be written as

$$[A]\left\{\frac{\partial Y}{\partial X_k}\right\} = \{RHS\} \tag{9}$$

where A and RHS are the matrix of coefficients and the right-hand-side vector, respectively, defined in Eq. (8). Since the equations are linear, the derivatives of their solution with respect to the elements of the matrix A may be obtained by substituting Eq. (9) for F in Eq. (2) and, then, writing the corresponding sensitivity equations using the differentation pattern of Eq. (3):

$$[A]\frac{\partial}{\partial A_{ij}}\left\{\frac{\partial Y}{\partial X_k}\right\} = -\left[\frac{\partial A}{\partial A_{ij}}\right]\left\{\frac{\partial Y}{\partial X_k}\right\} \tag{10}$$

In this set of equations, the matrix $\partial A/\partial A_{ij}$ is all empty except unity at the location occupied by the element A_{ij} in the matrix A. The vector of the partial derivatives of Y with respect to X_k is available from the solution of Eq. (9), and the derivatives of the RHS in Eq. (9) with respect to A_{ij} are null so they do not appear in Eq. (10). Consequently, the unknown derivatives $\partial(\partial Y/\partial X_k)/\partial A_{ij}$ may be obtained by back-substitution of the new right-hand-side vector over the matrix A decomposed once in the solution of Eq. (9) and saved.

These derivatives measure the influence of the partial sensitivity derivatives A_{ij} on the sensitivity of the system with respect to X and may be adopted as indicators of the strength of the couplings among the parts of the system. A full survey of the coupling strengths would require solution of Eqs. (9) and (10) for each combination of A_{ij} and X_k. In the case of Y and X expressed in nonhomogeneous physical units, a dimensionless form shown in the Appendix for Eq. (8) would have to be used to obtain the coupling strength indicators that could be compared with each other. The comparison would be useful to identify relatively weak couplings that might be dropped from the system's mathematical model. Thus, the coupling strength indicators may augment the system analyst's judgment in searching for a compromise between the system-model simiplicity and its predictive accuracy.

Conclusions

The paper addresses the problem of the sensitivity of a complex, internally coupled system behavior (response) to changes in independent variables. It is assumed that the system analysis is made up of self-contained analyses, corresponding to disciplines and/or physical subsystems, which exchange input/output data with each other. The problem is solved by formulating rigorous sensitivity equations—the global sensitivity equations—derived in a form based on the residuals of the governing equations, and in a new form that does not rely on the residuals. The latter is judged to be more useful to engineers than the former. It allows evaluation of system sensitivity to independent variables on the basis of the partial derivative information obtained locally within each contributing engineering discipline, or within each physical subsystem analysis, consistent, with the decomposition of the design process among the specialty groups, and compatible with the technology of distributed computing. The equations eliminate the need for costly and potentially inaccurate finite differencing performed on the entire system analysis, and are capable of accepting experimentally obtained sensitivity data. Their computational cost advantage over the reference finite-difference procedure increases with the number of self-contained analyses into which the system analysis can be partitioned, and is reduced proportionally to the volume of the coupling information.

Derivatives of the solution of the sensitivity equations with respect to the equation coefficients may be useful as indicators of the strength of the couplings among the parts of the system. Ranking these indicators by their magnitudes may identify weak couplings that might be eliminated from the system's model to make it simpler without significant loss of its predictive accuracy.

The global sensitivity equations in either form are offered as a tool to support the design process by contributing the system sensitivity information as an aid for human judgment and/or for use in formal optimization. Inasmuch as the global sensitivity equations answer quantitatively, with the first-order of the accuracy, the "what if" questions underlying the design process, they may be regarded as a first-order mathematical model of that process.

Appendix

Alternative derivation, generalization to a case of N CA's, and a dimensionless format for the global sensitivity equations, Eqs. (8), are discussed below.

Alternative Derivation

An alternative derivation begins with Eq. (5), differentiated with respect to one particular independent variable, say, X_k. Using the chain rule, the derivatives of Y are

$$\partial Y_\alpha/\partial X_k = (\partial f_\alpha/\partial X_k) + (\partial f_\alpha/\partial Y_\beta)(\partial Y_\beta/\partial X_k) + (\partial f_\alpha/\partial Y_\gamma)(\partial Y_\gamma/\partial X_k) \tag{A1a}$$

$$\partial Y_\beta/\partial X_k = (\partial f_\beta/\partial X_k) + (\partial f_\beta/\partial Y_\alpha)(\partial Y_\alpha/\partial X_k) + (\partial f_\beta/\partial Y_\gamma)(\partial Y_\gamma/\partial X_k) \tag{A1b}$$

$$\partial Y_\gamma/\partial X_k = (\partial f_\gamma/\partial X_k) + (\partial f_\gamma/\partial Y_\alpha)(\partial Y_\alpha/\partial X_k) + (\partial f_\gamma/\partial Y_\beta)(\partial Y_\beta/\partial X_k) \tag{A1c}$$

Collecting the given and unknown terms, and rearranging, yields the GSE2 in the format of Eq. (8).

Possible Singularity or Ill-Conditioning of the Matrix of Coefficients in Eqs. (7) and GSEx

Geometrically speaking, a solution to Eqs. (1), if nonlinearities are present, is a point in hyperspace common to all of the hypersurfaces corresponding to the equations involved. At that point, the hypersurfaces remain in one of the following three geometrical relations to each other: 1) they intersect; 2) they are tangential; and 3) in some dimensions they intersect and in some they are tangential. Since Eqs. (7) are a linear approximation of Eqs. (1) in the neighborhood of the solution point, they are represented by a set of hyperplanes. In case 1, these hyperplanes intersect at the solution point so that Eqs. (7) are nonsingular. Then, it follows that the linear sensitivity equations, Eq. (8), derived from Eqs. (7) are nonsingular too. However, if some of the hyperplane intersection angles are too acute, Eqs. (7) and, correspondingly, Eq. (8) may be ill-conditioned. Finally, in cases 2 and 3 some of the hyperplanes will coincide (overlap) and that will render Eqs. (7) and, consequently, Eq. (8) singular. Similar reasoning and conclusion apply also to Eq. (4). A more detailed discussion of these abnormal conditions is given in Ref. 1, Appendix A.

GSE1 and GSE2 Generalized to a Case of N CA's

A simple extrapolation of the pattern from the case of three CA's discussed in conjunction with Eqs. (1), (4), and (8), leads to Eqs. (1), GSE1 and GSE2, respectively, taking on the form of Eqs. (A2), (A3), and (A4):

$$A_i[(X, \ldots Y_j \ldots Y_l \ldots),\ Y_l] = 0 \qquad i = 1 \rightarrow N \quad j \neq i; l \neq i \tag{A2}$$

$$[B] \cdot \{\partial Y_i / \partial X_k\} = -\{\partial A_i / \partial X_k\} \tag{A3}$$

$$B_{ij} = \partial A_i / \partial Y_j$$

$$[B] \cdot \{\partial Y / \partial X_k\} = \{\partial f_i / \partial X_k\} \tag{A4}$$

$$B_{ij} = -\partial f_i / \partial Y_j;\ B_{ii} = I$$

Dimensionless Form

To transform the GSE2 from the form of Eq. (8) to a dimensionless form, we normalize the variables Y and X in Eq. (1) to unity by dividing them by their initial values (if any of them is zero, a suitable nonzero value is used instead). The normalization yields, showing two partial derivatives as examples,

$$(\partial \tilde{Y}_{\alpha i} / \partial \tilde{Y}_{\beta j}) = (\partial Y_{\alpha i} / \partial Y_{\beta j})(Y^o_{\beta j} / Y^o_{\alpha i}) \tag{A5}$$

$$(\partial \tilde{Y}_{\alpha i} / \partial \tilde{X}_k) = (\partial Y_{\alpha i} / \partial X_k)(X^o_k / Y^o_{\alpha i}) \tag{A6}$$

This normalization introduced into Eqs. (5-7) leads to a dimensionless form for Eq. (8). For instance, the partition β of Eq. (8) becomes

$$-\frac{\partial f_\beta}{\partial Y_\alpha} l_{\beta\alpha ij} \quad l \quad -\frac{\partial f_\beta}{\partial Y_\gamma} l_{\beta\gamma ij} \quad \frac{\partial Y_\beta}{\partial X_k} l_{\beta xk} \quad \frac{\partial F_\beta}{\partial X_k} l_{\beta xk} \tag{A7}$$

where

$$l_{\beta\alpha ij} = Y^o_{\alpha i} / Y_{\beta j}, \qquad l_{\beta\gamma ij} = Y^o_{\gamma i} / Y_{\beta j}, \qquad l_{\beta xk} = X^o_k / Y^o_{\beta j}$$

An example of a nondimensional version of the GSE2 obtained from the dimensional version by means of Eqs. (A6) is given in Table 2. It shows elimination of the great numerical disparity of the terms in the dimensional version of the GSE2.

Acknowledgments

The author gratefully acknowledges interaction with several colleagues, in particular: J. Barthelemy, G. Giles, R. Haftka, P. Hajela, E. Haug, B. James, S. Padula, L. Schmit, and R. Tolson, whose suggestions contributed to this study.

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